PRIFYSGOL ABERYSTWYTH UNIVERSITY

EXAMINATION FOR ENTRANCE SCHOLARSHIPS JANUARY 2019

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables are not required.

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Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\left|\frac{i+1}{3-2i}\right| \sqrt{\frac{13}{2}}$ [3 marks]

(b) $(\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot (\mathbf{i}-\mathbf{j}+\mathbf{k})$ [2 marks]

(c) $\det\left(\left(\begin{smallmatrix} 2 & 3 \\ 5 & 2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -2 \\ -1 & 3 \end{smallmatrix}\right)\right)$ [3 marks]

2. (a) Differentiate $\sin(\pi x(x^2 + \frac{3}{4}))$ with respect to x. [3 marks]

(b) Calculate $\int_0^{\frac{1}{2}} (4x^2+1)\cos(\pi x^3 + \frac{3\pi}{4}x)dx$ using the result of (a). [5 marks]

3. (a) A complex number $z = x + iy$ satisfying $|z-1| = 2|z+1|$ is represented by the point $P(x, y)$ in an Argand diagram. Show that the locus of P is a circle, find its radius and centre, and sketch it. [7 marks]

(b) Let $p(z)$ be a quadratic polynomial with real coefficients such that the two roots z_1 and z_2 of the equation $p(z) = 0$ are non-real.

(i) Assume that the modulus of z_1 is 1. Explain why the modulus of z_2 must also be 1. [2 marks]

(ii) Assume that z_1 lies in the locus considered in (a). Explain why z_2 must also lie in this locus. [2 marks]

(iii) In addition to the assumptions in (i) and (ii), suppose that $p(0) = 5$. Determine the values of a, b, c such that $p(z) = az^2 + bz + c$. [6 marks]

4. The plane Π contains the origin and is perpendicular to the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(a) Show that the points $(1, 1, 1)$ and $(3, 2, 4)$ lie in the plane Π. [3 marks]

(b) Write down the vector equation of the line L passing through the point $(1, 2, 3)$ and parallel to $\mathbf{v}$. [2 marks]

(c) Find the coordinates of the point of intersection of L and Π. [3 marks]

5. (a) Find the 2×2 matrices representing the following transformations in the plane:

(1) Anticlockwise rotation by $\frac{\pi}{3}$ radians about the origin. [3 marks]

(2) Reflection in the line $x = 0$. [2 marks]

(3) Reflection in the line $y = \sqrt{3}x$. [3 marks]

(b) The transformation T consists of (1) followed by (2) followed by (3). Calculate the 2×2 matrix representing T. [3 marks]

6. Consider the function $f(x) = \sqrt{3}\sin x - \cos x - \sqrt{2}\,x$, where x is in **radians**.

(a) Find the values of a, b, c such that $f'(x) = a\cos(x-b) + c$. [5 marks]

(b) Find all the stationary points of $f(x)$ in the interval $0 \le x \le 8$. [3 marks]

Section B

7. (a) Prove using mathematical induction that for all positive integers n,

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

[9 marks]

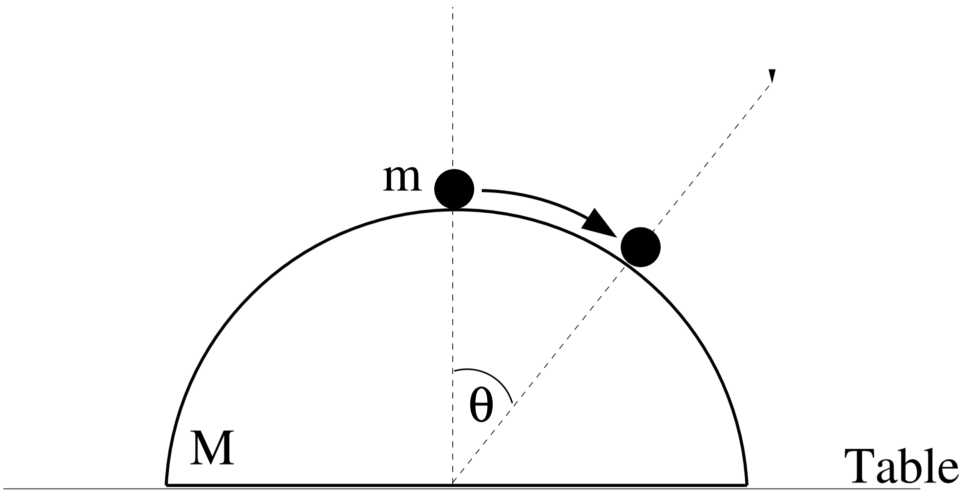
(b) Let $f(x) = 1 + \frac{4}{5}x^3$. Use the formula in (a) to find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{5k}{n}\right)\frac{5}{n}.$$

[8 marks]

(c) Explain why (b) constitutes a calculation of $\int_0^5 f(x)dx$ from first principles.
[3 marks]

8. A point particle of mass m sits at rest on top of a frictionless hemisphere of radius R and mass M. The hemisphere also rests on a frictionless table, as shown below. The particle is given a tiny kick and slides down the hemisphere to an angle θ as measured from the top of the hemisphere.



(a) Let v_x and v_y be the horizontal and vertical velocities of the point particle. Let V_x be the velocity of the hemisphere. Using conservation of momentum show that $v_x = \frac{M}{m} V_x$. Hence deduce that

$$v_y = \tan\theta \left(1 + \frac{m}{M}\right) v_x.$$

[4 marks]

(b) Using conservation of energy, and the results from part (a), show that horizontal velocity of the particle is given by,

$$v_x^2 = \frac{2gR(1-\cos\theta)}{\left(1+\frac{m}{M}\right)\left(1+\left(1+\frac{m}{M}\right)\tan^2\theta\right)}.$$

[5 marks]

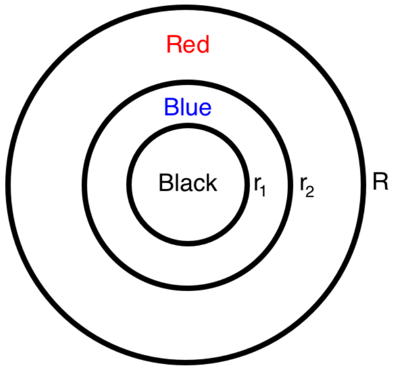
(c) Assuming $m = M$, show that the angle at which the particle loses contact with the hemisphere is given by the following condition,

$$\cos^3\theta - 6\cos\theta + 4 = 0.$$

Hint: the particle loses contact when its horizontal velocity is a maximum.
[8 marks]

(d) Given that the expression in part (c) has $\cos\theta - 2$ as a factor, find the angle at which the particle loses contact with the hemisphere. [3 marks]

9. Consider a circular target with three sections (black, blue and red). Suppose the target has a radius of R, and the black circle in the centre is of radius r_1, and the boundary between the blue and the red sections is at radius r_2 (see picture below).



(a) Suppose that a particular archer definitely hits the target from a distance of 40 yards and her arrows are distributed uniformly over the target. Find the radii, r_1 and r_2 in terms of R, if the probabilities $P(Black)$, $P(Blue)$ and $P(Red)$ are in the ratios 1:2:3. [8 marks]

(b) Again assuming the archer always hits the target, the scores for red, blue and black are k, $2k$ and $3k$ respectively. Find k such that the expected score for a particular arrow is 7. [4 marks]

(c) Suppose at a distance of 60 yards the probability that the archer hits the target reduces to 0.8. The scores for red, blue and black are now 1, 2 and 3 respectively.

(i) Complete the following table for the score with a single arrow:

x	$P(S = x)$
0	
1	
2	
3	

[4 marks]

(ii) Suppose the archer now fires three arrows from 60 yards. Each arrow is independent of the other two, with a probability of 0.8 of hitting the target and uniformly distributed on the target if it hits. If S_1, S_2 and S_3 are the scores obtained with the three arrows, find the expectation of $R = S_1 + S_2 + S_3$. [4 marks]