PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE EXAMINATION
JANUARY 2020

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers should be written in the answer books provided, including any diagrams, graphs or sketches. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.
- Statistical tables will be provided. Note that the tables refer to the **right-hand** tails of the distributions, that is, probabilities of the form $p = \mathbb{P}(X \geq x)$ where X is a random variable and x an **upper** percentage point of its distribution.

Information

- 2-D rotations and reflections are represented by matrices as follows.

Anticlockwise rotation through angle ϕ about the origin : $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

Reflection in the line $y = (\tan\phi)\,x$: $\begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}$

Section A

1. Simplify the following expressions as far as possible, showing your working.

(a) $5\,\frac{1-i}{1-2i} - 3$ [3 marks]

(b) $(2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ [2 marks]

(c) $\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}^{-1} - 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ [5 marks]

2. A complex number $z = x + iy$ satisfying $|z - 3| = |2z + i|$ is represented by the point $P(x, y)$ in an Argand diagram. Show that the locus of P is a circle, and find its radius and centre. [8 marks]

3. Given that $-2 + i$ is a root of the cubic equation

$$x^3 - x^2 - 15x - 25 = 0,$$

find the other two roots, explaining your method for each root. [4 marks]

4. In the following, a is a positive real number.

(a) Differentiate $\ln(x(x + a)^2)$ with respect to x. [4 marks]

(b) Using the result of (a), simplify

$$\int_a^{2a} \frac{3x^2 + 4ax + a^2}{x^3 + 2ax^2 + a^2x} dx$$

as far as possible. [4 marks]

5. Use mathematical induction to prove that

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

for all positive integers n. [7 marks]

6. The plane Π has equation $x + y + z = 0$. The point P lies on the plane Π and has coordinates $(1, 2, a)$. The line L is normal to Π and passes through P.

(a) Find the value of a. [1 mark]

(b) Write down the vector equation of the line L. [3 marks]

(c) Points A and B lie on L, both at distance 3 from P, on the opposite sides of the plane Π. Find the coordinates of A and B. [5 marks]

(d) Find (in radians) the acute angle between the line OA and the plane Π. [4 marks]

7. Consider the following transformations in the plane:

R_1 is the anticlockwise rotation by $\frac{2\pi}{3}$ radians about the origin.

R_2 is the reflection in the line $x = \sqrt{3}y$.

(a) Find the 2×2-matrices representing R_1 and R_2. [4 marks]

(b) The transformation T_1 consists of R_1 followed by R_2, while T_2 consists of R_2 followed by R_1. Find the matrices representing T_1 and T_2. [4 marks]

(c) Show that T_1 and T_2 in (b) are reflections, and find the lines in which they reflect. [2 marks]

Section B

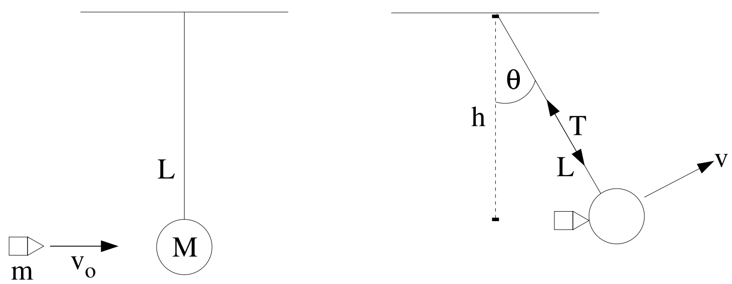
8. Consider the function $f(x) = (\cos x - 3)\sin x + 2x$ where x is in **radians**.

(a) Show that $f'(x) = a\cos^2 x + b\cos x + c$, where the constants a, b, c are to be found. [4 marks]

(b) Find all the stationary points of $f(x)$ in the interval $1 \le x \le 7$, and determine their nature. [12 marks]

(c) Explain why the equation $f(x) = 0$ has *exactly one* root in the interval $1 \le x \le 7$, and use a calculator to find an approximate value of the root to the accuracy of 1 decimal place. [4 marks]

9. A small wooden block with mass M is suspended from the lower end of a light cord of length L. The block is initially at rest. A bullet with mass m is fired at the block with a horizontal velocity v_0. The bullet strikes the block and becomes embedded in it.



(a) After the collision the combined object swings on the end of the cord. Consider the object at the moment when it reaches vertical distance h from the attachment point of the cord (see figure).

(i) Write down the component of the gravitational force acting on the object in the direction of the cord, in terms of M, m, h, L, and g (the acceleration due to gravity). [4 marks]

(ii) Hence show that velocity of the object is given by

$$v = \sqrt{\frac{L}{(M+m)}\left(T - (M+m)g\frac{h}{L}\right)},$$

where T is the magnitude of the tension force in the cord. [5 marks]

(b) By applying conservation of energy, show that the velocity of the combined object *immediately after* the collision is given by $V = \sqrt{2g(L-h)+v^2}$. [5 marks]

(c) By applying conservation of momentum, find an expression for the initial velocity v_0 of the bullet in terms of the quantities appearing in the expression for v above. [4 marks]

(d) Given that $h = 0.8$ metres, $M = 0.8$ kilograms, $m = 0.012$ kilograms, $L = 1.6$ metres and $T = 4.8$ newtons, find the value of v_0. (Use $g = 9.8\frac{\text{m}}{\text{s}^2}$.) [2 marks]

10. (a) Ecologists wish to test the theory that the numbers of eggs found in woodpeckers' nests can be modelled by a Poisson distribution with mean 1.6. The numbers of eggs in each of 50 nests were recorded as follows:

	0	1	2	3 or more
Frequency	15	16	7	12

(i) State suitable hypotheses for a goodness of fit test. [1 mark]

(ii) Carry out a χ^2 goodness of fit test on this data set, using a 5% level of significance, and state your conclusion. [9 marks]

(b) In an industrial process making steel wire, faults occur randomly at a rate of 1.5 per 1000 metres of wire.

(i) Find the probability that exactly 2 faults occur in 1500 metres of wire. [4 marks]

(ii) Suppose that there is a fault after 750 metres of wire production. Let L (in metres) be the random variable corresponding to the length of wire produced before the next fault. Calculate $P(L > x)$ and hence find the probability density, $f(x)$, of L. [6 marks]