PRIFYSGOL ABERYSTWYTH UNIVERSITY

ENTRANCE EXAMINATION
FEBRUARY 2023

FURTHER MATHEMATICS

Time allowed: 1 hour 30 minutes

- All answers (including any diagrams, graphs or sketches) should be written on paper, and scanned into a **single** PDF file. Graph paper is not required.
- Answer **all** questions in Section A and **two** questions from Section B.
- Candidates are permitted to use calculators, provided they comply with A level examining board regulations. They must be made available on request for inspection by invigilators, who are authorised to remove any suspect calculators.

Information and relevant formulas

- Powers of complex numbers: $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
- Statistical tables will be provided. Note that the tables refer to the **right-hand** tails of the distributions, that is, probabilities of the form $p = \mathbb{P}(X \geq x)$ where X is a random variable and x an **upper** percentage point of its distribution.
- Formulas related to standard distributions (e.g. for probability, mean, and variance) can be found on the back page of the statistical tables.

Section A

1. Simplify the following expressions as far as possible, showing your workings.

(a) $\dfrac{4i+3}{(3-i)^2}$; [4 marks]

(b) $\left|2(\mathbf{i}-5\mathbf{j}+\mathbf{k})+(\mathbf{i}+2\mathbf{j}-\mathbf{k})+(\mathbf{i}+5\mathbf{j}-\mathbf{k})\right|$; [4 marks]

(c) $\left(\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}^{-1}-\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}^{-1}\right)^{-1}$. [6 marks]

2. Solve the equation $5z-4\,\overline{z}=3\overline{z}+8-i+7z$, for the complex number z. (Here $\overline{z}$ is the complex conjugate of z.) [6 marks]

3. Determine the value of a such that the matrix $\begin{pmatrix}7 & -1\\ 4 & a\end{pmatrix}$ has no inverse. [3 marks]

4. Consider the complex number $z=\frac{1}{10}(3i+4)$.

(a) Express z in the form $r(\cos\theta+i\sin\theta)$ where $r>0$ and $-\pi\leq\theta\leq\pi$ (in radians to 3dp). [4 marks]

(b) Represent z and z^2 as points in an Argand diagram. [3 marks]

(c) Find the smallest positive integer n for which the imaginary part of z^n is negative, explaining your reasoning. [4 marks]

(d) Find the smallest positive integer n for which $|z^n|<10^{-4}$, explaining your reasoning. [3 marks]

5. The complex number z satisfying $2|z-2|=|3z-1+i|$ is represented by the point $P(x,y)$ in an Argand diagram. Find the equation of the locus of P in terms of x and y, and interpret it geometrically. [8 marks]

6. Consider the polynomial $f(x)=5x^3+13x^2+9x+c$, where c is a constant.

(a) Determine the value of c such that $f(\frac{2}{5})=0$. [2 marks]

(b) For the value of c found in (a), find all the roots of the equation $f(x)=0$, explaining your method for each root. [6 marks]

7. Prove by mathematical induction that

$$\begin{pmatrix}3 & -2\\ -2 & 3\end{pmatrix}^n=\frac{1}{2}\begin{pmatrix}1+5^n & 1-5^n\\ 1-5^n & 1+5^n\end{pmatrix},$$

for all positive integers n. [7 marks]

Section B

8. Plane Π has equation $x + 2y + 3z = 6$. Line L_1 passes through point $A(2, 3, 4)$, is perpendicular to Π, and intersects Π at point B. Furthermore, point C has coordinates $(2, 2, 0)$, and line L_2 has vector equation $\mathbf{r} = 3\mathbf{j} + \lambda(\mathbf{j} - 2\mathbf{i})$.

(a) Write down the equation of L_1 in the Cartesian form. [3 marks]

(b) Find the coordinates of B. [4 marks]

(c) Show that (i) C lies on L_2, and (ii) L_2 lies on Π. [4 marks]

(d) Find the angles and the lengths of the sides of the triangle ABC. [9 marks]

9. The position vector $\mathbf{x}$ (metres) at time t seconds of an object of mass M is

$$\mathbf{x} = \sin(3t)\,\mathbf{i} + \cos(3t)\,\mathbf{j} + e^{-2t}\,\mathbf{k}.$$

The initial kinetic energy of the object (at $t = 0$) is 13 J.

(a) Find (to 2dp) the time at which the object passes through the plane $z = 0.5$, and state the coordinates of the crossing point (to 2dp). [4 marks]

(b) Describe (in a few words) the nature of the motion of the object for large values of t, justifying your reasoning. [3 marks]

(c) Find expressions for the velocity and acceleration vectors $\mathbf{v}$ (ms^{-1}) and $\mathbf{a}$ (ms^{-2}) of the object at time t seconds. [4 marks]

(d) Calculate the mass M, and hence find the kinetic energy of the object at time t seconds. [5 marks]

(e) Find, in vector form, the force acting on the object at time t seconds. [2 marks]

(f) Calculate (to 2dp) the work done by the force acting on the object during the first second (i.e. from $t = 0$ to $t = 1$ second). [2 marks]

10. (a) At a particular point on a cycle path, passing cyclists are counted during 10-minute observation intervals. For each interval, the number of passing cyclists can be modelled by a Poisson distribution.

(i) Based on past records, the average number of cyclists passing in 10 minutes is 11. Find (to 4dp) the probability that exactly 3 cyclists pass the point in a 10-minute interval. [2 marks]

(ii) Given the past average in (i), find (to 4dp) the probability that at least 14 cyclists pass in a 10-minute interval. [2 marks]

(iii) In order to boost the usage of the cycle path, some improvement work was done. Let p be the probability that at least 14 cyclist pass the point in a 10-minute interval, after the improvement. The passes were counted in five distinct 10-minute intervals; X is the number of those intervals in which at least 14 cyclists pass. The result was $X = 4$.

Using X as your test statistic, carry out a hypothesis test to assess evidence that p exceeds the past value found in (ii). State clearly the hypotheses, and the distribution of X assuming the null hypothesis. Calculate the p-value and state your conclusion using a 1% significance level. [8 marks]

(b) In a psychology experiment, researchers are interested in establishing whether rats show preferences for particular routes through a maze. In the experiment, 100 rats were allowed to choose a route through the maze. The numbers of rats taking each of the four possible routes are shown in the table below:

Route	1	2	3	4
Number of rats	23	22	30	25

Carry out a χ^2 test at the 10% significance level to establish whether there is evidence of a preference for particular routes. Make sure to state the null hypothesis, the degree of freedom, the critical value for this test, and the formula for the statistic you calculate. [8 marks]